A NOTE ON FROBENIUS-SCHUR INDICATORS

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Abstract. This exposition concerns two different notions of Frobenius-Schur indicators for finite-dimensional Hopf algebras. These two versions of indicators coincide when the underlying Hopf algebra is semisimple. We are particularly interested in the family of pivotal finite-dimensional Hopf algebras with unique pivotal element; both indicators are gauge invariants of this family of Hopf algebras. We obtain a formula for the (pivotal) Frobenius-Schur indicators for the regular representation of a pivotal Hopf algebra. In particular, we use this formula for the four dimensional Sweedler algebra and demonstrate the difference of these two indicators.

1. Introduction

The second Frobenius-Schur indicator of a representation over $\mathbb{C}$ of a finite group was introduced more than a century ago [6]. This indicator $\nu_2(V)$ of an irreducible representation V of a finite group G can only be 0, 1 or -1 which is respectively determined by whether V is complex, real or pseudo-real representation of G. Moreover, the indicator $\nu_2(V)$ can be computed by the formula:

$$\nu_2(V) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g^2), \tag{1}$$

where χ_V is the character afforded by V. In general, the n-th Frobenius-Schur indicator of a representation of V of G is given by

$$\nu_n(V) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g^n)\,. \tag{2}$$

The higher indicators appear to be more obscure than the second indicator but they certainly carries the arithmetic properties of the group G. If one defines $\theta_n(g)$ for each $g\in G$ as the number of solutions $x\in G$ such that $x^n = g$, then θ_n is a class function on G. Moreover,

$$\theta_n = \sum_{V\in \mathrm{Irr}(G)} \nu_n(V)\chi_V$$

where $\mathrm{Irr}(G)$ denotes a complete set of non-isomorphic irreducible representations of G. The reader are referred to some basic references (such as [2, 8, 19]) on representation theory of finite groups for these basic facts.

A *bialgebra* H over a field $\Bbbk$ is a $\Bbbk$-algebra equipped with two algebra maps $\Delta : H \to H\otimes H$ and $\epsilon : H \to \Bbbk$ which satisfy the conditions:

$$(\mathrm{id}\otimes\Delta)\Delta = (\Delta\otimes \mathrm{id})\Delta, \quad (\epsilon\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\epsilon)\Delta = \mathrm{id}\ . \tag{3}$$

The Sweedler notation is often used to denote $\Delta(h) = h_{(1)}\otimes h_{(2)}$ with the summation suppressed. In the sequel, we will use the notation: $\Delta^{(1)} = \mathrm{id}_H$, $\Delta^{(2)} = \Delta$ and

$$\Delta^{(n+1)} = (\Delta\otimes \mathrm{id}_H)\Delta^{(n)} \quad \text{for all integer } n \geq 2.$$

In Sweedler's notation, $\Delta^{(n)}(h) = h_{(1)} \otimes \cdots \otimes h_{(n)}$ for all integer $n \geq 2$. The n-th Sweedler power $h^{[n]}$ of $h \in H$ is defined as

$$h^{[n]} = h_{(1)} \cdot \cdots \cdot h_{(n)},$$

and the assignment $P_n : H \to H$, $h \mapsto h^{[n]}$ will be called the n-th Sweedler power map of H.

The bialgebra H is called a *Hopf algebra* if H admits an *antipode* which is a $\Bbbk$-linear map $S : H \to H$ such that

$$m(S \otimes \mathrm{id})(h) = \epsilon(h) = m(S \otimes \mathrm{id})(h)$$

for all $h \in H$ where m denotes the multiplication on H. A *left (resp. right) integral* of H is a non-zero element $\Lambda \in H$ such that $h\Lambda = \epsilon(h)\Lambda$ (resp. $\Lambda h = \epsilon(h)\Lambda$). An element of H is called a *two-sided integral* if it is both left and right integral. We refer the readers to [13, 20] for the basic facts and notations for Hopf algebras.

The left (or right) integrals of a finite-dimensional Hopf algebra H span a 1-dimensional ideal. Moreover, H is semisimple if, and only if, $\epsilon(\Lambda)$ for some left integral Λ. In this case, Λ is a two sided integral and the normalized (two-sided) integral Λ, that is $\epsilon(\Lambda) = 1$, of H is uniquely determined by H.

In the remainder of this note, we simply consider finite-dimensional Hopf algebras over the field of complex numbers $\mathbb{C}$.

The group algebra $\mathbb{C}[G]$ is a semisimple Hopf algebra with the comultiplication map $\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$, the counit map $\epsilon : \mathbb{C}[G] \to \mathbb{C}$ and the antipode S given by

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} \tag{4}$$

for $g \in G$. The element $\Lambda = \frac{1}{|G|} \sum_{g \in G} g$ is a normalized integral of $\mathbb{C}[G]$, i.e. $\epsilon(\Lambda) = 1$.

Linchenko and Montgomery [11] generalized the notion of the n-th Frobenius-Schur indicator $\nu_n(V)$ for the representation V of semisimple Hopf algebra H over $\mathbb{C}$ by using uniqueness of normalized integral semisimple Hopf algebras, namely for each integer $n \geq 2$,

$$\nu_n(V) = \chi_V(\Lambda^{[n]})\,. \tag{5}$$

where χ_V is the character afforded by V.

The Frobenius-Schur theorem for the second indicator in relation to the existence of non-degenerate H-invariant symmetric or skew-symmetric bilinear form on a irreducible representation of a complex semisimple Hopf algebra was discovered in [11]. The arithmetic properties of the higher indicators were extensively studied in [10]. One of the remarkable consequences of these arithmetic properties is the Cauchy theorem [10] for complex semisimple Hopf algebras H which asserts that $\dim H$ and $\exp(H)$ have common prime factor, where $\exp(H)$ can be defined as the order of Drinfeld element in the Drinfeld double $D(H)$ (cf. [3]). The result also answered the questions raised by Etingof and Gelaki in [3].

Before the inception of [11], some notions of Frobenius-Schur (FS) indicator were also considered in rational conformal field theory (RCFT). Bantay introduced a version of the 2nd FS indicator for RCFT as a formula in terms of the modular data [1], and a categorical version of the 2nd FS indicator was studied by Fuchs, Ganchev, Szlachnyi, and Vescernys in [7]. These apparently unrelated notions of Frobenius-Schur indicators suggested a possibility that they are related and are invariant of the underlying monoidal categories. The author in collaboration Mason [12] introduced a definition for the representations of semisimple quasi-Hopf algebras H which is a generalization of the definition of Linchenko-Montgomery.

Moreover, if H and K are complex semisimple quasi-Hopf algebra with monoidally equivalent representation categories via the monoidal equivalence $\mathcal{F} : \mathrm{Rep}(H) \to \mathrm{Rep}(K)$, then $\nu_2(V) = \nu_2(\mathcal{F}(V))$.

When V is a simple H-module, $\nu_2(V)$ can only be 0, 1, or -1. A version of Frobenius-Schur Theorem is proved for semisimple quasi-Hopf algebra with this generalized notion of indicator. The canonical pivotal structure discovered in [5] has played an very important role in [12]. This fact was also recognized by Schauenburg, and he reintroduced the definition of FS indicator for a representation of a complex semisimple quasi-Hopf algebra using the canonical pivotal structure in [18]. This indicator is shown to be identical to that of [12]. The discovery initiated a generalized definition of the n-th Frobenius-Schur indicator for each object in a pivotal tensor category [15]. By using the canonical pivotal structures in a modular category and in the representation category of a complex semisimple quasi-Hopf algebra, the Bantay's 2-nd indicator formula and the indicators for semisimple (quasi-)Hopf algebras are recovered from this notion of (pivotal) Frobenius-Schur indicators (cf. [16, 14]). These new developments on indicators have yielded a Cauchy Theorem for semisimple quasi-Hopf algebras[14, Theorem 8.4], and two Congruence Subgroup Theorems for modular categories [17, Theorems 6.7, 6.8].

2. Pivotal Hopf algebras and beyond

Let H be a finite-dimensional Hopf algebra over $\mathbb{C}$ but not necessarily semisimple. Then $\mathrm{Rep}(H)$ is a pivotal category if, and only if, the square of the antipode S of H is a conjugation by a group-like element g, i.e.$S^2(h) = ghg^{-1}$. Such group-like element is called a *pivotal element* and H is called a *pivotal Hopf algebra*. It is clear that if g is a pivotal element, then the coset $gZG(H)$ is the set of all pivotal elements of H where $ZG(H)$ is the group of central group-like elements of H. Unlike the semisimple case, there is no canonical choice of pivotal element in general. It is important to note that the pivotal Frobenius-Schur indicators are invariant of pivotal categories. If one would like distinguish only the monoidal structures of the representation categories of two non-isomorphic pivotal Hopf algebras, using the higher pivotal Frobenius-Schur indicators for these pivotal categories might not be straightforward.

This suggests a need of invariants for the monoidal structure of $\mathrm{Rep}(H)$ for any finite-dimensional Hopf algebras H. By [16, Theorem 2.2], if H and K are finite-dimensional Hopf algebra over $\mathbb{C}$ such that $\mathrm{Rep}(H)$ and $\mathrm{Rep}(K)$ are $\mathbb{C}$-linearly equivalent monoidal categories, then $H \cong K^F$ where K^F is a Drinfeld twist by a gauge transformation F on H which satisfies some 2-cocycle conditions. We simply call H and K are *gauge equivalent* Hopf algebras in this case.

Let $\mathfrak{C}$ be a collection of finite-dimensional Hopf algebras over $\mathbb{C}$ which is closed under gauge equivalence classes. Following the terminology of [9], a quantity $f(H)$ defined for each $H \in \mathfrak{C}$ is called a gauge invariant for the collection $\mathfrak{C}$ if $f(K) = f(H)$ for all Hopf algebras K gauge equivalent to H. Typical examples of gauge invariants of H are $\dim(H)$ and quasi-exponent $\mathrm{qexp}(H)$ defined by Etingof and Gelaki [4], and they are gauge invariants for the collection $\mathfrak{H}$ of all the finite-dimensional Hopf algebras over $\mathbb{C}$.

In [9], the author, in collaboration with Kashina and Montgomery, has introduced a $\mathbb{C}$-linear recursive sequence $\nu(H) = \{\nu_n(H)\}_{n\geq 1}$ which is given by

$$\nu_1(H) = 1, \quad \nu_n(H) = \mathrm{Tr}(S \circ P_{n-1}) \quad \text{for } n \geq 2,$$

where S, P_m are respectively the antipode and the m-th Sweedler power of H. It has been shown in [9, Theorem 2.2] that the sequence $\nu(H)$ is a gauge invariant for $\mathfrak{H}$. Moreover, by [9, Corollary 2.6], we have

$$\nu_n(H) = \lambda(\Lambda^{[n]}) \tag{6}$$

where λ and Λ are both left (or both right) integrals of H^* and H respectively satisfying the condition $\lambda(\Lambda) = 1$. When H is semisimple, it follows by [10] that $\nu_n(H)$ coincides with the n-th Frobenius-Schur indicator of the regular representation of a semisimple Hopf algebra defined in [11].

Now, we come back to consider the collection $\mathfrak{P}$ of pivotal Hopf algebras with a unique pivotal element such as the Taft algebras or the small quantum group $U_q(\mathfrak{sl}_2)$. Since the pivotal element is unique for $H \in \mathfrak{P}$, $\mathrm{Rep}(H)$ has exactly one pivotal structure. If K is a Hopf algebra gauge equivalent to H, then $\mathrm{Rep}(K)$ also has a unique pivotal structure and so K has a unique pivotal element. Therefore, $\mathfrak{P}$ is a collection closed under gauge equivalence. Moreover, for any monoidal equivalence $\mathcal{F} : \mathrm{Rep}(H) \to \mathrm{Rep}(K)$ for some Hopf algebras $H, K \in \mathfrak{P}$, $\mathcal{F}$ is automatically a pivotal equivalence. Therefore, by the preceding remark,

$$\nu_n^p(H) = \nu_n^p(\mathcal{F}(H)) = \nu_n^p(K)$$

for all $n \in \mathbb{N}$, where $\nu_n^p(V)$ denotes the n-th pivotal Frobenius-Schur indicator of V in the pivotal category $\mathrm{Rep}(H)$. In particular, the sequence $\nu^p(H) = \{\nu_n^p(H)\}_{n\geq 1}$ is a gauge invariant for $\mathfrak{P}$.

One natural question is how these gauge invariants $\nu(H)$ and $\nu^p(H)$ are related. In the following section, we will demonstrate that these two sequences are different, in general, for $H \in \mathfrak{P}$.

3. Pivotal Frobenius-Schur indicators for pivotal Hopf algebras

Let H be a pivotal Hopf algebra with a pivotal element g. For any left H-module V, the map $j : V \to V^{\vee\vee}$ given by

$$j(x) = g\hat{x}$$

is a natural isomorphism of H-modules, where $\hat{x}(f) = f(x)$ for all $f \in H^*$. Since g is group-like, $j : Id \to (-)^{\vee\vee}$ is an isomorphism of tensor functors. Hence, j defines a pivotal structure on $\mathrm{Rep}(H)$. Consider the pivotal category $(\mathrm{Rep}(H), j)$. Let λ and Λ be a right integral of H^* and a left integral of H, respectively, such that $\lambda(\Lambda) = 1$. For any $V \in \mathrm{Rep}(H)$, $\theta : H \otimes {}^\circ V \to H \otimes V$ defined by $\theta(h \otimes v) = h_{(1)} \otimes h_{(2)}v$ is an H-module isomorphisms, where ${}^\circ V$ denotes the trivial left H-module with the underlying space V. Hence,

$$(H \otimes V)^H = \Lambda_{(1)} \otimes \Lambda_{(2)}V.$$

In particular, the linear map $\alpha : V \to (H \otimes V)^H$, $v \mapsto \Lambda_{(1)} \otimes \Lambda_{(2)}v$, is an isomorphism with

$$\alpha^{-1}(\sum_i h_i \otimes v_i) = \sum_i \lambda(h_i)v_i\,.$$

Thus that map $\overline{\alpha} : V \to \mathrm{Hom}_H(\mathbb{C}, H \otimes V)$, defined by

$$\overline{\alpha}(v)(1) = \Lambda_{(1)} \otimes \Lambda_{(2)}v,$$

is a $\mathbb{C}$-linear isomorphism.

By [15], the map $E_{H,V} : \mathrm{Hom}_H(\mathbb{C}, H \otimes V) \to \mathrm{Hom}_H(\mathbb{C}, V \otimes H)$ is defined by

$$E_{H,V}(f) := \left(\mathbb{C} \xrightarrow{\mathrm{db}} H^\vee \otimes H^{\vee\vee} \xrightarrow{H^\vee \otimes f \otimes H^{\vee\vee}} H^\vee \otimes H \otimes V \otimes H^{\vee\vee} \xrightarrow{\mathrm{ev} \otimes j^{-1}} V \otimes H\right),$$

where $\mathrm{db} : \mathbb{C} \to H^\vee \otimes H^{\vee\vee}$ is the dual basis map. If $f \in \mathrm{Hom}_H(\mathbb{C}, H \otimes V)$ and $f(1) = \sum_i h_i \otimes v_i$, then $E_{H,V}(f)(1) = \sum_i v_i \otimes g^{-1} h_i$.

Now, we take $V = H^{\otimes(m-1)}$ for some integer $m \geq 1$. Note that $H^{\otimes 0} = \mathbb{C}$ by convention. The isomorphism $\overline{\alpha} : V \to \mathrm{Hom}_H(\mathbb{C}, H \otimes V)$ determines a unique endomorphism $\overline{E}^{(m)} = \overline{\alpha}^{-1} \circ E_{H,H^{\otimes(m-1)}} \circ \overline{\alpha}$ on V. Since $E_{H,\mathbb{C}} = \mathrm{id}$, and so is $\overline{E}^{(1)}$. For $m \geq 2$, $h \in H$ and $v \in H^{\otimes(m-1)}$,

$$\overline{E}^{(m)}(h \otimes v) = \lambda(\Lambda_{(2)} h)\Lambda_{(3)} v \otimes g^{-1}\Lambda_{(1)}. \tag{7}$$

The m-th pivotal Frobenius-Schur indicator $\nu_m^p(H)$ of the regular representation of H is defined as $\mathrm{Tr}(E_{H,H^{\otimes(m-1)}})$, which is also equal to $\mathrm{Tr}(\overline{E}^{(m)})$. The following lemma is useful for simplifying $\mathrm{Tr}(\overline{E}^{(m)})$.

Lemma 3.1. *Let A be an algebra over a field $\Bbbk$, $a_1, \ldots, a_n \in A$, $f \in A^*$. Let $T : A^{\otimes n} \to A^{\otimes n}$ be the $\Bbbk$-linear map given by*

$$T(v_1 \otimes \cdots \otimes v_n) = f(v_1) a_1 v_2 \otimes \cdots \otimes a_{n-1} v_n \otimes a_n, \quad \text{for } v_1 \otimes \cdots v_n \in A^{\otimes n}.$$

Then $\mathrm{Tr}(T) = f(a_1 \cdots a_{n-1} a_n)$.

Proof. Let $\{b^i\}$, $\{b_i\}$ be dual bases for A^* and A respectively. Then we have

$$\begin{aligned}
\mathrm{Tr}(T) &= \sum_{i_1,\cdots,i_n} f(b_{i_1})\langle b^{i_1}, a_1 b_{i_2}\rangle \cdots \langle b^{i_{n-1}}, a_{n-1} b_{i_n}\rangle \langle b^{i_n}, a_n\rangle \\
&= \sum_{i_1,\cdots,i_n} f(b_{i_1})\langle b^{i_1}, a_1 b_{i_2}\rangle \cdots \langle b^{i_{n-1}}, a_{n-1} a_n\rangle \\
&= \sum_{i_1} f(b_{i_1})\langle b^{i_1}, a_1 \cdots a_{n-1} a_n\rangle \\
&= f(a_1 \cdots a_{n-1} a_n). \quad \square
\end{aligned}$$

Therefore, we have

Proposition 3.2. *Let H be a finite-dimensional pivotal Hopf algebra over $\mathbb{C}$. With respect to the pivotal element $g \in H$, $\nu_1^p(H) = 1$ and the m-th pivotal Frobenius-Schur indicator $\nu_m^p(H)$ of H is given by*

$$\nu_m^p(H) = \lambda(\Lambda_{(2)}^{[m-1]} g^{-1} \Lambda_{(1)}).$$

for all integers $m \geq 2$.

Proof. By definition, $\nu_1^p(H) = \mathrm{Tr}(E_{H,\mathbb{C}}) = \mathrm{Tr}\left(\mathrm{id}_{\mathrm{Hom}_H(\mathbb{C},H)}\right)$. Since $\dim \mathrm{Hom}_H(\mathbb{C}, H) = 1$, $\nu_1^p(H) = 1$.

We now assume $m \geq 2$. By the preceding remark, $\nu_m^p(H) = \mathrm{Tr}(\overline{E}^{(m)})$. In view of (7) and Lemma 3.1,

$$\nu_m^p(H) = \lambda(\Lambda_{(2)}\Lambda_{(3)} \cdots \Lambda_{(m)} g^{-1} \Lambda_{(1)}) = \lambda(\Lambda_{(2)}^{[m-1]} g^{-1} \Lambda_{(1)}).$$

$\square$

We can now use the proposition to compute the pivotal Frobenius-Schur indicators of a Taft algebra [21] which has exactly one pivotal element. Hence, they are Hopf algebras in $\mathfrak{P}$. For simplicity, we consider the Taft algebra of dimension 4, which is also called the Sweedler algebra H_4.

A Taft algebra $T_n(\omega)$ is a $\mathbb{C}$-algebra generated by x, g subject to the relations:

$$xg = \omega gx, \quad x^n = 0, \quad g^n = 1$$

where ω is a primitive n-th root of unity. The comultiplication Δ, the counit ϵ and the antipode of $T_n(\omega)$ are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \epsilon(g) = 1, \quad \epsilon(x) = 0,$$
$$S(g) = g^{-1} \quad \text{and} \quad S(x) = -gx\,.$$

In particular, $S^2(h) = ghg^{-1}$ for all $h \in T_n(\omega)$. Therefore, g is a pivotal element. Since there is no non-trivial central group-like element in $T_n(\omega)$, g is the unique pivotal element of $T_n(\omega)$, i.e. $T_n(\omega) \in \mathfrak{P}$.

For the Sweedler algebra H_4 or $T_2(-1)$, $\Lambda = x + gx$ is a left integral and $\lambda \in H^*$, defined by $\lambda(g^i x^j) = \delta_{i,0}\delta_{j,1}$ for $0 \le i, j \le 1$, is a right integral such that $\lambda(\Lambda) = 1$. Note that

$$\Delta^{(m)}(g) = g^{\otimes m}, \quad \Delta^{(m)}(x) = x \otimes 1^{\otimes(m-1)} + g \otimes x \otimes 1^{\otimes(m-2)} + \cdots + g^{\otimes(m-1)} \otimes x$$

for all integer $m \ge 2$. Therefore,

$$g^{[m]} = g^m, \quad x^{[m]} = \sum_{i=0}^{m-1} g^i x, \quad (gx)^{[m]} = \sum_{i=0}^{m-1} gxg^{m-i-1}.$$

For any integer $m \ge 2$,

$$\begin{aligned}
\nu_m^p(H_4) &= \lambda(\Lambda_{(2)}^{[m-1]} g^{-1} \Lambda_{(1)}) \\
&= \lambda(1^{[m-1]}gx + x^{[m-1]} + g^{[m-1]}x + (gx)^{[m-1]}g) \\
&= \lambda\left(\sum_{i=0}^{m-2} g^i x + g^{m-1}x + \sum_{i=0}^{m-2} (-1)^{m-i-1} g^{m-i} x\right) \\
&= \#\{i \mid 0 \le i \le m-1, i \text{ even}\} - \#\{i \mid 0 \le i \le m-2,\, m-i \text{ even}\} \\
&= \begin{cases} 0 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}. \end{cases}
\end{aligned}$$

Therefore, $\nu^p(H_4) = \{1, 0, 1, 0, \dots\}$.

Another version of gauge invariant $\nu(H_4)$, simply called the FS-indicators of H_4, was computed in [9] and it is given by

$$\nu(H_4) = \{1, 2, 3, 4, \dots\}.$$

Obviously, $\nu^p(H_4) \ne \nu(H_4)$ are both invariant of the monoidal category $\mathrm{Rep}(H_4)$.

The example H_4 suggests the notion of Frobenius-Schur indicators defined in [9] and the pivotal Frobenius-Schur indicators defined in [15] are intrinsically different. One natural question is whether there exists a gauge invariants $\kappa(H)$ for $H \in \mathfrak{H}$ such that $\kappa(H) = \nu^p(H)$ for all $H \in \mathfrak{P}$. It is highly unclear whether there is any natural relationships (in terms of some structures of $\mathrm{Rep}(H)$) between $\nu^p(H)$ and $\nu(H)$ for $H \in \mathfrak{P}$. Such relations could reveal more interesting gauge invariants for the complete collection $\mathfrak{H}$ of finite-dimensional Hopf algebras.

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